

ON THIN CARPETS FOR DOUBLING MEASURES

CHANGHAO CHEN, SHENGYOU WEN

ABSTRACT. We study subsets of \mathbb{R}^d which are thin for doubling measures or isotropic doubling measures. We show that any subset of \mathbb{R}^d with Hausdorff dimension less than or equal to $d - 1$ is thin for isotropic doubling measures. We also prove that a self-affine set that satisfies *OSCH* (open set condition with holes) is thin for isotropic doubling measures. For doubling measures, we prove that Barański carpets are thin for doubling measures.

Keywords Doubling measures, Isotropic doubling measures, Barański carpets.

1. INTRODUCTION

A Borel regular measure μ on metric space X is called *doubling* if there is a constant $C \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty,$$

for any $x \in X$ and $0 < r < \infty$. We call C the *doubling constant* of μ . Denoted by $\mathcal{D}(X)$ all the doubling measures on X . A closely related concept with doubling measures is a *doubling metric space*. A metric space is called doubling metric space if there exist positive integer N such that any ball of radius r can be covered by a collection of N balls of radius $r/2$. It's easy to see that $\mathcal{D}(X) \neq \emptyset$ implies X is doubling. On the other hand, if the space X is doubling and complete, then $\mathcal{D}(X) \neq \emptyset$, for more details see [10, 13, 26, 30]. A subset E of X is called *thin* for doubling measures if $\mu(E) = 0$ for every $\mu \in \mathcal{D}(X)$. Being thin for *isotropic doubling measures* is defined analogously. In this paper we are going to investigate some subsets of \mathbb{R}^d are thin for doubling measures or isotropic doubling measures. First we recall a useful estimate for doubling measures, see [8, Chapter 13] [29].

Lemma 1.1. *Let $\mu \in \mathcal{D}(\mathbb{R}^d)$ and Q_1, Q_2 be two cubes with $Q_1 \subset Q_2$. Then*

$$C^{-1} \left(\frac{|Q_1|}{|Q_2|} \right)^\beta \leq \frac{\mu(Q_1)}{\mu(Q_2)} \leq C \left(\frac{|Q_1|}{|Q_2|} \right)^\alpha, \quad (1.1)$$

Date: February 6, 2015.

The first author was supported by the Vilho, Yrjö, and Kalle Väisälä foundation.

where $|E|$ means the diameter of E and C, α, β are positive constant which only depend on μ .

Lemma 1.1 implies that every subset E of \mathbb{R}^d with Hausdorff dimension zero is thin for doubling measures (the same argument as mass distribution principle, see [5, Chapter 4]). Various examples of thin sets for doubling measures relate to the concept of porosity, see [9, 21, 23, 24, 29]. Doubling measures give zero weight to any smooth hyper-surface, see [25, p.40]. But for every $d \geq 2$, there exist rectifiable curve in \mathbb{R}^d which is not thin, see [7]. In [27] the authors asked that: *Is the graph of continuous function thin for doubling measures?* This question was negatively answered in [22]. We will show that rectifiable curves and graphs of continuous function are thin for isotropic doubling measures. The following definition is from [12].

Definition 1.2. *A Borel measure μ on \mathbb{R}^d is isotropic doubling if there is a constant $A \geq 1$ such that*

$$A^{-1} \leq \frac{\mu(R_1)}{\mu(R_2)} \leq A,$$

whenever R_1 and R_2 are congruent rectangular boxes with nonempty intersection.

We denote by $\mathcal{ID}(\mathbb{R}^d)$ all isotropic doubling measures on \mathbb{R}^d . Isotropic doubling measures arise from the study of δ -monotone mappings. We refer to [12] for more details about isotropic doubling measures and δ -monotone mappings. In [12] they proved that isotropic doubling measures are absolutely continuous to \mathcal{H}^{d-1} (Hausdorff measure) and for every $d \geq 2$, there exists an isotropic doubling measure on \mathbb{R}^d which is singular with respect to the Lebesgue measure. The following question of [12] arises naturally. *Is it true that every isotropic doubling measure on \mathbb{R}^d , $d \geq 2$, is absolutely continuous with respect to the s -dimensional Hausdorff measure for all $s < d$?* This question was one of the motivations of this work. We don't know the answer. However by applying a similar estimate as Lemma 1.1 to isotropic doubling measures, we get the following result.

Proposition 1.3. *Let $E \subset \mathbb{R}^d$ with $\dim_H E \leq d - 1$, then E is thin for isotropic doubling measures on \mathbb{R}^d .*

Motivated by the above question of [12], we consider the *self-affine sets*. By adding the condition *OSCH* (see Definition 3.1) on self-affine sets, we have the following result.

Theorem 1.4. *A self-affine set that satisfies OSCH is thin for isotropic doubling measures.*

For the doubling measures, things become more complicated. So we consider a special class of *self-affine carpets* on the plane. Barański [1] generalized the construction of Bedford-McMullen carpets to build a class of self-affine carpets. We call them Barański carpets, see Definition 3.6 or [1]. For Bedford-McMullen carpets, see [2, 5, 20]. For Barański carpets, we have the following result.

Theorem 1.5. *Barański carpets are thin for doubling measures.*

Acknowledgement: We would like to thank Tuomo Ojala and Ville Suomala for reading the manuscript and giving many helpful comments.

2. ISOTROPIC DOUBLING MEASURES

We start from a useful lemma of [12].

Lemma 2.1. *Let μ be a isotropic doubling measure on \mathbb{R}^d , $d \geq 2$ with doubling constant A . Then: (i) For any congruent rectangular boxes $R_1, R_2 \subset \mathbb{R}^d$.*

$$A^{-m} \leq \frac{\mu(R_1)}{\mu(R_2)} \leq A^m, \quad (2.1)$$

where $m = \lceil \frac{\text{dist}(R_1, R_2)}{\text{diam} R_1} \rceil + 1$.

(ii) Let $Q \subset \mathbb{R}^d$ be a cube, and let F be a face of Q . The pushforward $\pi_\# \mu_Q$ of μ_Q under the orthogonal projection $\pi : Q \rightarrow F$ is comparable to \mathcal{L}_F^{d-1} with constants that depend only on d and A .

The following is an analogue of Lemma 1.1 for isotropic doubling measures.

Lemma 2.2. *Let $\mu \in \mathcal{ID}([0, 1]^d)$, $d \geq 2$ with the doubling constant A and I be a cube in $[0, 1]^{d-1}$, and J be interval in $[0, 1]$. Then*

$$C^{-1}|I|^{d-1}|J|^\beta \leq \mu(I \times J) \leq C|I|^{d-1}|J|^\alpha, \quad (2.2)$$

where α, β and C are positive constants depending only on d and A .

Proof. Let $\nu(E) := \mu(I \times E)$ for $E \subset [0, 1]$. Then ν is a doubling measure on $[0, 1]$ with the doubling constant A . Applying Lemma 1.1 to ν , we have

$$C_1^{-1}|J|^\beta \leq \frac{\nu(J)}{\nu([0, 1])} \leq C_1|J|^\alpha, \quad (2.3)$$

where C_1, α , and β are positive constants depending only on A . Applying the second part of Lemma 2.1, we see that $\nu([0, 1])$ is comparable to $|I|^{d-1}$ with the constant depending on d and A only. Thus we have finished the proof. \square

Applying Lemma 2.2 and the same argument as mass distribution principle (see [5, Chapter 4]), we arrive at the following corollary immediately.

Corollary 2.3. *Let $\mu \in \mathcal{ID}([0, 1]^d)$, then there exist a positive constant α which only depends on μ , such that μ is absolutely continuous to $\mathcal{H}^{d-1+\alpha}$.*

Proof of Theorem 1.3. Let $\mu \in \mathcal{ID}([0, 1]^d)$. If $d = 1$, then μ is doubling measure on $[0, 1]$ and E has Hausdorff dimension zero. By Lemma 1.1, we know that any set with Hausdorff dimension zero is thin for doubling measures. Thus we arrive at the result for $d = 1$.

For the case $d \geq 2$, applying the Corollary 2.3, there is positive α such that μ is absolutely continuous to $\mathcal{H}^{d-1+\alpha}$. Since $\dim_H E \leq d - 1$, so $\mathcal{H}^{d-1+\alpha}(E) = 0$ and thus $\mu(E) = 0$. We complete the proof by the arbitrary choice of $\mu \in \mathcal{ID}([0, 1]^d)$. \square

Since any k -rectifiable sets of \mathbb{R}^d ([19, Chapter 15]) have Hausdorff dimension k , for $k < d$ they are thin for isotropic doubling measures on \mathbb{R}^d . Let $f : [0, 1] \rightarrow \mathbb{R}$ be a function. Recall that the graph of function f is $G(f) := \{(x, f(x)) : x \in [0, 1]\}$. Now we are going to apply Lemma 2.2 to prove that the graphs of continuous functions are thin for isotropic doubling measures.

Proposition 2.4. *Let $f : [0, 1]^d \rightarrow [0, 1]$ be a continuous function. Then $\mu(G(f)) = 0$ for all $\mu \in \mathcal{ID}([0, 1]^{d+1})$.*

Proof. Let $\mu \in \mathcal{ID}([0, 1]^{d+1})$, then there is positive C and α such that the estimate (2.2) holds. Since f is continuous on $[0, 1]^d$, it's well know that f is uniformly continuous on $[0, 1]^d$. Thus for any $\epsilon > 0$, there is δ such that $|f(x) - f(y)| \leq \delta$ for all $x, y \in [0, 1]^d$ with $|x - y| \leq \delta$.

Choose $n \in \mathbb{N}$, such that $2^{-n}\sqrt{d} \leq \delta$. Let \mathcal{D}_n denote all the dyadic cubes of $[0, 1]^d$ with side-length 2^{-n} . For each cube I of \mathcal{D}_n , there is an interval $I' \subset [0, 1]$ with $|I'| \leq \epsilon$ such that $\{(x, f(x)) : x \in I\} \subset I \times I'$. Whence

$$G(f) \subset \cup_{I \in \mathcal{D}_n} I \times I'.$$

By applying Lemma 2.2, we have

$$\mu(G(f)) \leq \sum_{I \in \mathcal{D}_n} \mu(I \times I') \leq \sum_{I \in \mathcal{D}_n} C\epsilon^\alpha |I|^d \leq C\epsilon^\alpha (\sqrt{d})^d. \quad (2.4)$$

Let $\epsilon \rightarrow 0$, then we have $\mu(G(f)) = 0$. We finish the proof by the arbitrary choice of μ . \square

By using the same idea (applying Lusin theorem) as in [22], the result of Proposition 2.4 is also holds if we change continuous function to measurable function.

Corollary 2.5. *Let $f : [0, 1]^d \rightarrow [0, 1]$ be a measurable function with respect to Lebesgue measure. Then $G(f)$ is thin for doubling measures on $[0, 1]^{d+1}$.*

Proof. Applying Lusin theorem (and it's normal corollary), for any $\epsilon > 0$, there is a continuous function $g : [0, 1]^d \rightarrow [0, 1]$ such that

$$\mathbb{L}^d(\{x : f(x) \neq g(x)\}) < \epsilon. \quad (2.5)$$

Let $\mu \in \mathcal{ID}([0, 1]^d)$. Lemma 2.1(ii) says that there is a constant C which depends on d and μ only, such that

$$\mu(A \times [0, 1]) \leq C\mathbb{L}^d(A) \text{ for any } A \subset [0, 1]^d. \quad (2.6)$$

Let $D = \{x : f(x) \neq g(x)\}$. Since

$$G(f) = G(g) \cup (G(f) \setminus G(g)) \subset G(g) \cup (D \times [0, 1]),$$

the estimate (2.6) and Proposition 2.4, we have $\mu(G(f)) \leq C\epsilon$. By the arbitrary choice of ϵ , we have $\mu(G(f)) = 0$. Thus $G(f)$ is thin for isotropic doubling measures on $[0, 1]^{d+1}$. \square

3. DOUBLING MEASURES AND SELF-AFFINE SETS

Let Λ be the attractor of the self-affine IFS

$$\mathcal{F} := \{f_i(x) = T_i x + t_i\}_{i=1}^m, x \in \mathbb{R}^d. \quad (3.1)$$

We always assume that the maps f_i are contractive and T_i are non-singular linear maps for each $1 \leq i \leq m$. For more details on self-affine sets, see [5, Chapter 9]. The following condition is often used to avoid overlap of IFS. Recall that the IFS \mathcal{F} is said to satisfy the open set condition (*OSC*) if there exists a non-empty open set $V \subset \mathbb{R}^d$ such that

- $f_i(V) \subset V$ holds for all $1 \leq i \leq m$;
- $f_i(V) \cap f_j(V) = \emptyset$ for all $i \neq j$.

We recall some standard notation for IFS. Let $S = \{1, \dots, m\}$. Denote $S^* := \cup_{k=1}^{\infty} S^k$ all the finite words and $S^{\mathbb{N}}$ all the infinite words. Let $\sigma = (i_1, \dots, i_k) \in S^k$. Define $f_\sigma := f_{i_1} \circ \dots \circ f_{i_k}$.

Definition 3.1. *We say that the IFS \mathcal{F} satisfies the *OSCH* (open set condition with hole) if \mathcal{F} satisfies the *OSC* and for the open set V , $V \setminus \bigcup_{i=1}^m f_i(V)$ has non empty interior.*

Definition 3.2. A map f is called *non-singular affine map* on \mathbb{R}^d , if there is a non-singular linear map T and a vector $t \in \mathbb{R}^d$ such that $f(x) = T(x) + t, x \in \mathbb{R}^d$. A map f is called *diagonal affine map* on \mathbb{R}^d , if there is a diagonal matrix $D = D(\lambda_1, \dots, \lambda_d)$ and a vector $t \in \mathbb{R}^d$ such that $f(x) = D(x) + t, x \in \mathbb{R}^d$. Note that the diagonal maps and non-singular linear maps coincide when $d = 1$. We call a cube Q *stable* if $Q \subset [0, 1]^d$ and there is $x \in \mathbb{R}^d$ and $\rho > 0$ such that $Q = x + \rho[0, 1]^d$.

Lemma 3.3. Let $Q \subset [0, 1]^d$ be stable cube and $\mu \in \mathcal{ID}(\mathbb{R}^d)$. Then there exist a positive constant C (depending on μ and side-length of Q) such that for any diagonal affine map f , we have

$$\mu(f(Q)) \geq C\mu(f([0, 1]^d)). \quad (3.2)$$

Proof. If $d = 1$, then we arrive at the estimate by applying the Lemma 1.1. Now we consider the case $d \geq 2$. We assume that $Q = I_1 \times \dots \times I_d$ where $I_i \subset [0, 1]$. Denote by a the side-length of Q and $V_Q = I_1 \times [0, 1]^{d-1}$. Let $\{Q_i\}_{i=1}^N$ be a sequence of closed cubes with the same edge length a and disjoint interior. Furthermore we ask that $\{Q_i\}_{i=1}^N$ satisfies $V_Q \subset \bigcup_{i=1}^N Q_i \subset [-2, 2]^d$ and for any Q_i and $Q_j, i \neq j$, there exist i_1, i_2, \dots, i_n such that $Q_i = Q_{i_1}, Q_j = Q_{i_n}$ and $Q_{i_k} \cap Q_{i_{k+1}} \neq \emptyset$ for $1 \leq k \leq n-1$. By a simple volume argument, $N \leq (\frac{4}{a})^d$. Thus

$$\mu(Q) \geq A^{-N}\mu(Q_i) \text{ for all } 1 \leq i \leq N. \quad (3.3)$$

Summing both sides over index i , we have

$$\mu(Q) \geq \frac{1}{NA^N}\mu(V_Q). \quad (3.4)$$

Let $C_1 = (\frac{a}{4})^d A^{-(\frac{4}{a})^d}$, then $\mu(Q) \geq C_1\mu(V_Q)$.

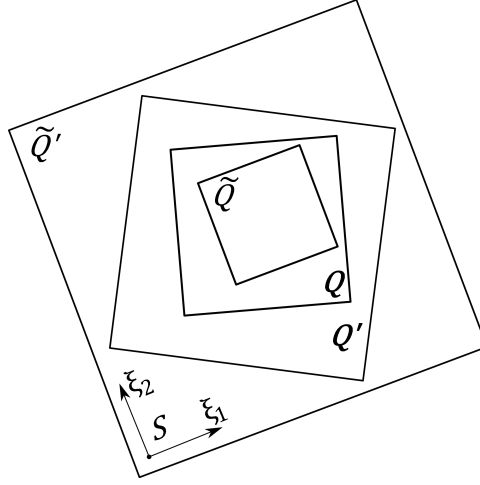
Since f is a diagonal map, we have that $f(Q_i)$ is a rectangle for $1 \leq i \leq N$. Applying the same argument as above, we have

$$\mu(f(Q)) \geq A^{-N}\mu(f(Q_i)) \text{ for all } 1 \leq i \leq N, \quad (3.5)$$

and $\mu(f(Q)) \geq C_1\mu(f(V_Q))$.

Applying the same argument to V_Q and $[0, 1]^d$ (in place of Q, V_Q), we have $\mu(f(V_Q)) \geq C_2\mu(f([0, 1]^d))$ where C_2 is a positive constant that depends on Q and the doubling constant A only. Letting $C = C_1C_2$ we complete the proof. \square

Now we are going to show that the above result also holds for any non-singular linear map. We will use the polar decomposition of a matrix. The polar decomposition says that for any matrix T , there exists a symmetric matrix S and orthogonal matrix O such that $T = OS$. Furthermore if T is non-singular, then S is positive definite. For more details see [4, Chapter 3].

FIGURE 1. The cubes of Proposition 3.4 for $d = 2$.

Proposition 3.4. *Let $Q \subset Q'$ and $\mu \in \mathcal{ID}(\mathbb{R}^d)$. Then for any non-singular affine map f , we have*

$$\mu(f(Q)) \geq C\mu(f(Q')), \quad (3.6)$$

where C is a positive constant doesn't depends on f .

Proof. If $d = 1$, then the non-singular map f is the same as diagonal map. Thus we arrive at the estimate by applying Lemma 1.1 again. Now we consider $d \geq 2$. For a non-singular map f , there is a non-singular matrix T and a vector $t \in \mathbb{R}^d$, such that $f(x) = Tx + t$ for $x \in \mathbb{R}^d$. For the convenience we use the same notations as above writing $T = OS$.

For positive definite matrix S , it's well known that there exist a standard orthogonal basis $\{\xi_1, \dots, \xi_d\}$ such that $S\xi_i = \lambda_i\xi_i$, and $\lambda_i > 0$ for all $1 \leq i \leq d$. Let $I(\xi_i) := \{t\xi_i : t \in [-1/2, 1/2]\}$, $1 \leq i \leq d$ and $Q_S = I(\xi_1) \times \dots \times I(\xi_d)$ be the unite cube. Let $\tilde{Q}(x, \rho) := x + \rho Q_S$. Denote by a the side-length of Q and x_0 the center of Q . By a simple geometric argument, we have

$$\tilde{Q}(x_0, \frac{a}{\sqrt{d}}) \subset B(x_0, \frac{a}{2}) \subset Q.$$

Denote by a' the side-length of Q' and x'_0 the center of Q' . Again by a simple geometric argument we have

$$Q' \subset B(x'_0, \frac{a'\sqrt{d}}{2}) \subset \tilde{Q}'(x'_0, a'\sqrt{d}).$$

Applying the same argument as in Lemma 3.3 to $\tilde{Q} \subset \tilde{Q}'$, there is a positive constant C such that

$$\mu(S\tilde{Q}) \geq C\mu(S\tilde{Q}'). \quad (3.7)$$

Note that the estimate (3.7) still holds with the same constant C after rotations and translations. Thus

$$\mu(f(\tilde{Q})) \geq C\mu(f(\tilde{Q}')). \quad (3.8)$$

This completes the proof since $f(\tilde{Q}) \subset f(Q)$ and $f(Q') \subset f(\tilde{Q}')$. \square

Proof of Theorem 1.4. Let Λ be the attractor of the self-affine IFS

$$\{f_i(x) = T_i x + t_i\}_{i=1}^m, x \in \mathbb{R}^d$$

which satisfies the *OSCH*. Since the IFS satisfies *OSCH*, there is an open set V and a cube Q with non empty interior such that $Q \subset V \setminus \bigcup_{i=1}^m f_i(V)$. It's well known that (see [5, Chapter 9]) there is compact cube Q' such that $f_i(Q') \subset Q', 1 \leq i \leq m, V \subset Q'$ and

$$\Lambda = \bigcap_{k=1}^{\infty} \bigcup_{\sigma \in S^k} f_{\sigma}(Q').$$

Since our IFS satisfies *OSCH* and by the position of Q , we have $f_{\sigma}(Q) \cap f_{\tau}(Q) = \emptyset$ for any $\sigma \neq \tau$ where $\sigma, \tau \in S^*$. For each $k \in \mathbb{N}$, denote $G_k = \bigcup_{\sigma \in S^k} f_{\sigma}(Q)$. Let $\mu \in \mathcal{ID}(\mathbb{R}^d)$. Note that $Q \subset Q'$ and f_{σ} is non-singular affine map for any $\sigma \in S^*$. Thus by Proposition 3.4, there exists a positive constant C such that

$$\mu(f_{\sigma}(Q)) \geq C\mu(f_{\sigma}(Q')), \text{ for any } \sigma \in S^*. \quad (3.9)$$

Since $f_{\sigma}(Q)$ are pair disjoint for $\sigma \in S^*$, we have

$$\mu(G_k) = \sum_{\sigma \in S^k} \mu(f_{\sigma}(Q)).$$

Summing two sides of equation (3.9) over $\sigma \in S^k$, we get

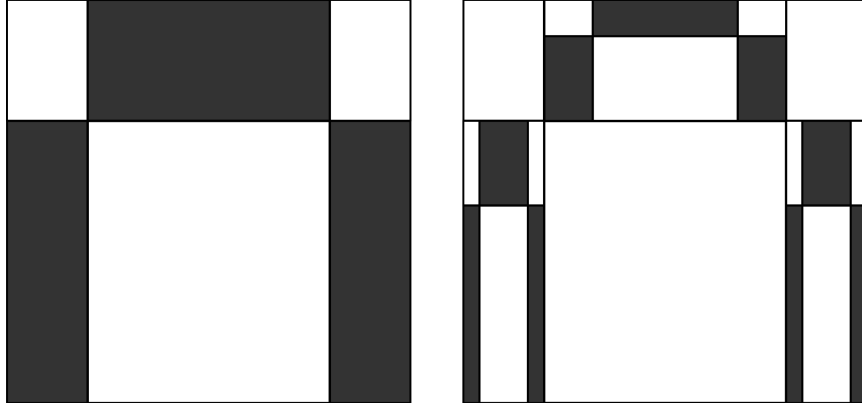
$$\mu(G_k) \geq C \sum_{\sigma \in S^k} \mu(f_{\sigma}(Q')) \geq C\mu(\Lambda), \quad (3.10)$$

where the last inequality holds since $\Lambda \subset \bigcup_{\sigma \in S^k} f_{\sigma}(Q')$.

Since $\bigcup_{k=1}^{\infty} G_k \subset Q'$, and $G_i \cap G_j = \emptyset$ for $i \neq j$, together with inequality (3.10), we have

$$\infty > \sum_{k=1}^{\infty} \mu(G_k) \geq C \sum_{k=1}^{\infty} \mu(\Lambda). \quad (3.11)$$

Thus we have $\mu(\Lambda) = 0$. \square

FIGURE 2. The first two steps in the construction of E .

We don't know whether Theorem 1.4 is also holds for doubling measures.

Question 3.5. *Is the attractor of IFS satisfies OSCH thin for doubling measures?*

Now we are going to prove Theorem 1.5. We first recall the construction of Barański carpets, see [1].

Definition 3.6. *Let $\{a_i\}_{i=1}^p, \{b_j\}_{j=1}^q$ be two sequences of positive numbers such that $\sum_{i=1}^p a_i = 1$ and $\sum_{j=1}^q b_j = 1$ where $p, q \in \mathbb{N}$ and $p, q \geq 2$. We have a partition of the unit square by q horizontal lines and p vertical lines. We exclude a sub-collection of these rectangles to form E_1 (We assume that at least one rectangle was excluded to avoid the trivial case). Iterate this construction for each rectangle of E_1 as above, in other words we replace each rectangle of E_1 by an affine copy of E_1 . In the end we have a limit set E . For an example see Figure 2. Recall that the limit set E is called BM (Bedford-McMullen) carpet if $a_i = 1/p$ for all $1 \leq i \leq p$ and $b_j = 1/q$ for all $1 \leq j \leq q$, see [5, chapter 9].*

Recall that a subset $E \subset \mathbb{R}^d$ is called porous if there exists $\alpha \in (0, 1)$, such that for any ball $B(x, r)$, there is a ball $B(y, \alpha r) \subset B(x, r)$ satisfies $B(y, \alpha r) \cap E = \emptyset$. The concept of porosity is closely related to the Assouad dimension. The connection is the following: A subset E of \mathbb{R}^d is porous if and only if $\dim_A E < d$. For more details, we refer to [17, Theorem 5.2]. It's well know that if a set $E \subset \mathbb{R}^d$ is porous, then E is thin for doubling measures on \mathbb{R}^d (by applying the density argument for doubling measure on the porosity set, the same as [25, p.40]). Since the Assouad dimension of Barański carpets can be less than or equal to 2, see [6, 18], thus we can't obtain Theorem 1.5 by apply the above

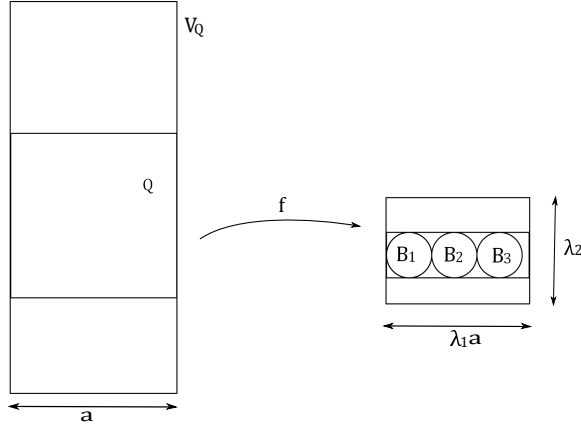


FIGURE 3.

mentioned result: if a set $E \subset \mathbb{R}^2$ has Assouad dimension less than 2, then E is porous, and so E is thin for doubling measures.

Lemma 3.7. *Let $Q \subset [0, 1]^2$ be a cube with edge length a and $\mu \in \mathcal{D}([0, 1]^2)$. Denoted by V_Q the smallest vertical strip of $[0, 1]^2$ which contains Q . Then there is a positive constant C depending on μ and a only, such that for any diagonal affine map $f(x) := D(\lambda_1, \lambda_2)(x) + t$ with $\lambda_1 \geq \lambda_2 > 0$, we have*

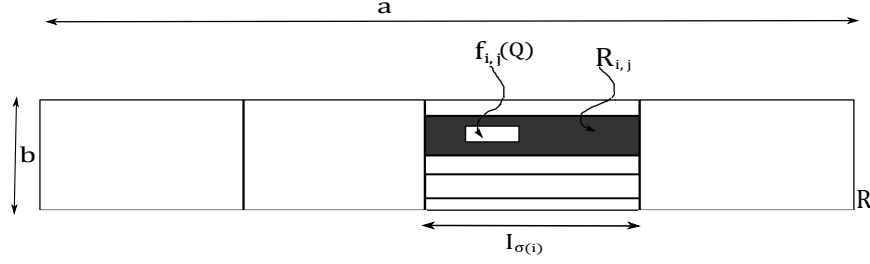
$$\mu(f(Q)) \geq C\mu(f(V_Q)). \quad (3.12)$$

Proof. Since f is diagonal map, $f(Q)$ is a rectangle with sides $a\lambda_1$ and $a\lambda_2$. We are going to place a sequence of closed balls with the same diameter $a\lambda_2$ inside the rectangle $f(Q)$. We put the first ball B_1 at the left part of $f(Q)$ and touching the left boundary of $f(Q)$. We put the second ball B_2 touching the first ball B_1 with disjoint interior. We continue to put the balls in the above way, see Figure 3. In the end we have $\lfloor \frac{\lambda_1}{\lambda_2} \rfloor$ balls inside $f(Q)$ where $\lfloor \frac{\lambda_1}{\lambda_2} \rfloor$ is the integer part of $\frac{\lambda_1}{\lambda_2}$. By a simple geometric estimate, we have that

$$f(V_Q) \subset \bigcup_{i=1}^{\lfloor \frac{\lambda_1}{\lambda_2} \rfloor} \frac{2}{a} B_i,$$

where $\rho B(x, r) := B(x, \rho r)$. Thus we have

$$\mu(f(V_Q)) \leq \sum_{i=1}^{\lfloor \frac{\lambda_1}{\lambda_2} \rfloor} \mu\left(\frac{2}{a} B_i\right) \leq A \sum_{i=1}^{\lfloor \frac{\lambda_1}{\lambda_2} \rfloor} \mu(B_i) \leq A\mu(f(Q)), \quad (3.13)$$

FIGURE 4. For each $R_{i,j}$ there is a hole $f_{i,j}(Q)$.

the second inequality holds by the doubling property of μ and the constant A depends on μ and a only. Thus we have completed the proof with $C = 1/A$. \square

Proof of Theorem 1.5. Let E be a Barański carpet constructed as above. For every $n \in \mathbb{N}$, let $R := I_a \times I_b$ be an n -level rectangle of E_n where I_a, I_b are subintervals of $[0, 1]$ with length a and b respectively. Without loss of generality we assume $a \geq b$. For the interval I_a , there exists a sequence of subintervals of I_a denoted by $\{I_{\sigma(i)}\}_{i=1}^{N(R)}$ where $\sigma(i) \in S^*$ and $N(R) \in \mathbb{N}$. These subintervals have the following properties (Moran cover),

- $I_a \subset \bigcup_i I_{\sigma(i)}$,
- $a_{\min}|I_{\sigma(i)}| < b \leq |I_{\sigma(i)}|$, $1 \leq i \leq N(R)$, where $a_{\min} = \min_{1 \leq i \leq p} \{a_i\}$.
- $\text{int}(I_{\sigma(i)}) \cap \text{int}(I_{\sigma(j)}) = \emptyset$ for $i \neq j$ where $\text{int}(I)$ means the interior of I .

Thus we may write $R = \bigcup_{i=1}^{N(R)} I_{\sigma(i)} \times I_b$. Let $R_i := I_{\sigma(i)} \times I_b$ and $I_{\sigma(i)} \times I_b = \bigcup_{j=1}^{q^{|\sigma(i)|}} R_{i,j}$ where $R_{i,j}$ is $(n + |\sigma(i)|)$ -level rectangle with $\text{int}(R_{i,j}) \cap \text{int}(R_{i,j'}) = \emptyset$ for $j \neq j'$. There is a cube $Q \subset [0, 1]^2$ such that $Q \cap E_1 = \emptyset$. We use the same notation V_Q as in Lemma 3.7. For each $R_{i,j}$, $1 \leq i \leq N(R)$, $1 \leq j \leq q^{|\sigma(i)|}$, denote by $f_{i,j}$ the affine map such that $f_{i,j}([0, 1]^2) = R_{i,j}$. See Figure 4. Denote

$$G(R) := \bigcup_{i=1}^{N(R)} \bigcup_{j=1}^{q^{|\sigma(i)|}} f_{i,j}(Q).$$

Let $\mu \in \mathcal{D}([0, 1]^2)$. By Lemma 3.7, there is a positive constant C_1 such that $\mu(f_{i,j}(Q)) \geq C_1 \mu(f_{i,j}(V_Q))$. Summing both sides over j to get

$$\sum_{j=1}^{q^{|\sigma(i)|}} \mu(f_{i,j}(Q)) \geq C_1 \sum_{j=1}^{q^{|\sigma(i)|}} \mu(f_{i,j}(V_Q)). \quad (3.14)$$

Let $\widetilde{R}_i := \bigcup_{j=1}^{q^{|\sigma(i)|}} f_{i,j}(V_Q)$. Notice that the side-length of \widetilde{R}_i are comparable with $|I_{\sigma(i)}|$ for each $1 \leq i \leq N(R)$. Thus there is a positive constant C_2 such that $\mu(\widetilde{R}_i) \geq C_2 \mu(R_i)$. Summing both sides over i , we have

$$\sum_{i=1}^{N(R)} \mu(\widetilde{R}_i) \geq C_2 \sum_{i=1}^{N(R)} \mu(R_i). \quad (3.15)$$

Combine the estimates (3.14) and (3.15), we arrive

$$\mu(G(R)) \geq C_1 C_2 \mu(R). \quad (3.16)$$

Let $G_n := \bigcup_{R \in E_n} G(R)$, then by estimate (3.16), we have that

$$\mu(G_n) \geq C_1 C_2 \mu(E_n) \geq C_1 C_2 \mu(E). \quad (3.17)$$

Given n_k , let $\widetilde{n}_k = \max\{N(R) : R \in E_n\}$ and $n_{k+1} = n_k + \widetilde{n}_k + 10$. Let $k = 1$, then we have a sequence n_k and G_{n_k} . By our choice of n_k and $Q \cap E_1 = \emptyset$, we observe that the sets G_{n_k} are pairwise disjoint subsets of $[0, 1]^2$. Applying estimate (3.17) to every G_{n_k} , we obtain

$$\infty > \mu\left(\bigcup_{k=1}^{\infty} G_{n_k}\right) \geq C_1 C_2 \sum_{k=1}^{\infty} \mu(E_{n_k}) \geq C_1 C_2 \sum_{k=1}^{\infty} \mu(E).$$

Thus we have $\mu(E) = 0$. We complete the proof by the arbitrary choice of $\mu \in \mathcal{D}([0, 1]^2)$. \square

3.1. Bedford-McMullen sponges in \mathbb{R}^3 . (Suggested by V. Suomala) Applying similar argument as in the proof of Theorem 1.5, we are going to prove that *Bedford-McMullen (BM) sponges* are thin for doubling measures on $[0, 1]^3$. We show the construction of *BM sponges* first. Let $p, q, u \in \mathbb{N}$ and $2 \leq p \leq q \leq u$. Divide $[0, 1]^3$ into $p \times q \times u$ rectangles of sides $1/p, 1/q$ and $1/u$. Select a subcollection of these rectangles to form E_1 . Iterate this construction in the usual way, with each rectangle replaced by an affine copy of E_1 , and let $E = \bigcap_{n \geq 1} E_n$ be the limiting set obtained. Let $Q \subset [0, 1]^3$ be a cube that $Q \cap E_1 = \emptyset$ and $V_Q := [0, 1]^3 \cap \pi_{xy}^{-1}(\pi_{xy}(Q))$ where π_{xy} is the orthogonal projection from \mathbb{R}^3 to plane \mathbb{R}_{xy} , here $\mathbb{R}_{xy} = \{(x, y, z) : z = 0\}$.

Proposition 3.8. *Let E be a BM sponge, then E is thin for doubling measures.*

Proof. Let R be a n -th rectangle of E_n . There is $n(R) \in \mathbb{N}$ such that

$$u^{-n} \leq p^{-n-n(R)} < u^{-n+1}.$$

Divide R (in the same way as the construction of E_1) $n(R)$ times into $(p \times q \times u)^{n(R)}$ rectangles. Let

$$I(R) := \{(i, j, k) : 1 \leq i \leq p^{n(R)}, 1 \leq j \leq q^{n(R)}, 1 \leq k \leq u^{n(R)}\}.$$

We may write $R = \cup_{\sigma \in I(R)} R_\sigma$. For $\sigma = (i, j, k)$ let $f_{i,j,k} := f_\sigma$. Denote $R_{i,j} = \cup_{k=1}^{u^{n(R)}} R_{i,j,k}$ and $R_i = \cup_{j=1}^{q^{n(R)}} R_{i,j}$. For each $R_\sigma, \sigma \in I(R)$, there is an affine map f_σ such that $R_\sigma = f_\sigma([0, 1]^3)$.

Denote $G(R) = \cup_{\sigma \in I(R)} f_\sigma(Q)$. Let $\mu \in \mathcal{D}([0, 1]^3)$. We are going to prove $\mu(G(R)) \gtrsim \mu(R)$ where \gtrsim means there is a constant C depends on μ only such that $\mu(G(R)) \geq C\mu(R)$. For the convenience in what follows we will use notation \gtrsim when there is a constant depending on μ only. Note that the constant may be different in different places. Applying the similar argument as in Lemma 3.7, it's not hard to see

$$\mu(f_\sigma(Q)) \gtrsim \mu(f_\sigma(V_Q)), \sigma \in I(R). \quad (3.18)$$

Thus

$$\sum_{k=1}^{u^{n(R)}} \mu(f_{i,j,k}(Q)) \gtrsim \sum_{k=1}^{u^{n(R)}} \mu(f_{i,j,k}(V_Q)) = \mu(V_Q(i, j)), \quad (3.19)$$

where $V_Q(i, j) := \cup_{k=1}^{u^{n(R)}} f_{i,j,k}(V_Q)$. Let $V'_Q(i, j) := R_{i,j} \cap \pi_{zx}^{-1}(\pi_{zx} V_Q(i, j))$. Applying the same argument as in Lemma 3.7 again, we get

$$\mu(V_Q(i, j)) \gtrsim \mu(V'_Q(i, j)) \quad (3.20)$$

and

$$\mu(\cup_{j=1}^{q^{n(R)}} V'_Q(i, j)) \gtrsim \mu(R_i). \quad (3.21)$$

Note that $f_\sigma(Q)$ are pair disjoint for $\sigma \in I(R)$. Combine the estimates (3.19), (3.20) and (3.21) we arrive

$$\mu(G(R)) \gtrsim \mu(R). \quad (3.22)$$

Let $G_n := \cup_{R \in E_n} G(R)$, then by estimate (3.22), we have that

$$\mu(G_n) \gtrsim \mu(E_n) \geq \mu(E). \quad (3.23)$$

Apply the same argument as in the proof of Theorem 1.5, we obtain the result. \square

Remark 3.9. *It can be believed that high dimensional Bedford-McMullen self-affine sets are thin for doubling measures. Note that Bedford-McMullen self-affine sets in $\mathbb{R}^d, d \geq 2$ satisfies OSCH.*

4. PURELY ATOMIC MEASURES

We say that a measure is *purely atomic*, if it has full measure on a countable set. For the results related purely atomic doubling measures, see [3, 11, 15, 16, 28]. It was asked in [15] whether there exist compact set $X \subset \mathbb{R}$ with positive Lebesgue measure so that all doubling measures μ on X are purely atomic. The answer is negative given by [3, 16]. In [3, 16], they proved that any compact set of \mathbb{R}^d with positive Lebesgue measure carries a doubling measure which is not purely atomic. We extend their result in the following way.

Proposition 4.1. *Let X be a closed subset of \mathbb{R}^d with positive Lebesgue measure, then every d -homogeneous measure on X is not purely atomic; furthermore, let $E \subset X$ and $L^d(E) > 0$, then $\mu(E) > 0$ for every d -homogeneous measure μ on X .*

A measure μ is called an s -homogeneous measure on X if there is a constant C such that for any $\lambda \geq 1$,

$$0 < \mu(B(x, \lambda r)) \leq C\lambda^s \mu(B(x, r)) < \infty.$$

Denote by $\mathcal{D}_s(X)$ all s -homogeneous measure on X . It's easy to see that $\mathcal{D}(X) = \bigcup_{s>0} \mathcal{D}_s(X)$. The s -homogeneous measures are related to s -homogeneous spaces, see [14, 26].

Proof of Proposition 4.1. Let $E \subset X$ and $L^d(E) > 0$. Let $\mu \in \mathcal{D}_d(X)$ (in [13] they proved that $\mathcal{D}_d(X) \neq \emptyset$). We are going to prove that $\mu(E) > 0$ (this implies that μ is not purely atomic).

We consider $B(x, r)$ as an open ball of metric space X (induced metric from \mathbb{R}^d) in the following for the convenience. Let $x_0 \in X$, then there exists n_0 such that $L^d(E \cap B(x_0, n_0)) > 0$. Since $\mu \in \mathcal{D}_d(X)$, there is constant C such that for any ball $B(x, r) \subset B(x_0, n_0)$, we have

$$\mu(B(x, n_0)) \leq C\left(\frac{n_0}{r}\right)^d \mu(B(x, r)). \quad (4.1)$$

Applying the doubling property of μ , we have that there is a constant C_1 such that $\mu(B(x_0, n_0)) \leq C_1 \mu(B(x, n_0))$ for any $x \in B(x_0, n_0)$. Thus there is a positive constant C_2 such that

$$\mu(B(x, r)) \geq C_2 r^d \text{ for any } B(x, r) \subset B(x_0, n_0). \quad (4.2)$$

It's well known that (4.2) implies (see [19, p.95])

$$\mu(A) \geq C_3 L^d(A) \text{ for any } A \subset B(x_0, n_0), \quad (4.3)$$

where C_3 is positive constant depends on μ, d, n_0 only. By the monotone property of μ and estimate (4.3), we have

$$\mu(E) \geq \mu(E \cap B(x_0, n_0)) \geq C_3 L^d(E \cap B(x_0, n_0)) > 0.$$

We complete the proof by the arbitrary choice of $\mu \in \mathcal{D}_d(X)$. \square

REFERENCES

- [1] K. Barański. Hausdorff dimension of the limit sets of some planar geometric constructions. *Adv. Math.*, 210(1):215-245, 2007.
- [2] T. Bedford. Crinkly curves, Markov partitions and dimension. PhD thesis, University of Warwick, 1984.
- [3] M. Csörnyei, V. Suomala, On Cantor sets and doubling measures. *J. Math. Anal. Appl.* 393 (2012), no. 2, 680-691.
- [4] L. C. Evans, R. F. Gariepy, Measure theory and fine properties of functions, *Studies in Advanced Mathematics*, CRC PRESS. 1993.
- [5] K. Falconer, *Fractal Geometry: Mathematical Foundation and Applications*, John Wiley, 1990.
- [6] Jonathan M. Fraser, Assouad type dimensions and homogeneity of fractals, *Trans. Amer. Math. Soc.* 366 (2014), no. 12, 6687-6733.
- [7] J. Garnett, R. Killip, R. Schul, A doubling measure on \mathbb{R}^d can charge a rectifiable curve, *Proc. Amer. Math. Soc.* 138(2010), 1673-1679.
- [8] J. Heinonen. *Lectures on analysis on metric spaces*. Springer-Verlag, New York, 2001.
- [9] D. Han, L. Wang, and S. Wen. Thickness and thinness of uniform Cantor sets for doubling measures. *Nonlinearity*, 22(2009), 545-551.
- [10] A. Käenmäki, T. Rajala, and V. Suomala: Existence of doubling measures via generalized nested cubes. *Proc. Amer. Math. Soc.* 140, 2012, 3275-3281.
- [11] R. Kaufman, J. M. Wu, Two problems on doubling measures, *Rev. Math. Iberoamericana*, 11(1995), 527-545.
- [12] L. V. Kovalev, D. Maldonado, and J.-M. Wu, Doubling measures, monotonicity, and quasiconformality, *Math. Z.* 257(2007), 525-545.
- [13] J. Luukkainen and E. Saksman. Every complete doubling metric space carries a doubling measure. *Proc. Amer. Math. Soc.*, 126 (1998) 531-534.
- [14] J. Luukkainen, Assouad dimension: antifractal metrization, porous sets, and homogeneous measures, *J. Korean Math. Soc.* 35(1998), 23-76.
- [15] M.L. Lou, S.Y. Wen, M. Wu, Two examples on atomic doubling measures, *J. Math. Anal. Appl.* 333 (2007) 1111-1118.
- [16] M. Lou, M. Wu, Doubling measures with doubling continuous part, *Proc. Amer. Math. Soc.* 138(2010), 3585-3589.
- [17] J. Luukkainen, Assouad dimension: antifractal metrization, porous sets, and homogeneous measures. *J. Korean Math. Soc.* 35 (1998), no. 1, 23-76.
- [18] John M. Mackay, Assouad dimension of self-affine carpets, *Conform. Geom. Dyn.* 15 (2011), 177-187.
- [19] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge University Press, Cambridge, 1995.
- [20] C. McMullen. The Hausdorff dimension of general Sierpiński carpets. *Nagoya Math. J.*, 96(1984), 1-9.
- [21] T. Ojala, On (α_n) -regular sets, *Ann. Acad. Sci. Fenn. Math.*, 39:655-673, 2014.
- [22] T. Ojala, T. Rajala, A function whose graph has positive doubling measure, arxiv.org/abs/1406.4693
- [23] T. Ojala, T. Rajala, V. Suomala, Thin and fat sets for doubling measures in metric spaces, *Studia Math.* 208 (2012), 195-211.

- [24] F.J. Peng and S.Y. Wen. Fatness and thinness of uniform cantor sets for doubling measures. *Sci. China Math.*, 54:7581, 2011.
- [25] E. M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton Mathematical Series, 43. Princeton University Press, Princeton, NJ, 1993.
- [26] A. L. Vol'berg and S. V. Konyagin. On measures with the doubling condition. *Izv. Akad. Nauk SSSR Ser. Mat.*, 51(3):666-675, 1987. MR903629 (88i:28006)
- [27] W. Wang, S. Wen and Z. Wen, Fat and thin sets for doubling measures in Euclidean space, *Ann. Acad. Sci. Fenn. Math.*, 38:535-546, 2013.
- [28] W. Wang, S. Wen, Z. Wen, Note on atomic doubling measures, quasisymmetrically thin sets and thick sets, *J. Math. Anal. Appl.* 385 (2012), 1027-1032.
- [29] J.-M. Wu, Null sets for doubling and dyadic doubling measures, *Ann. Acad. Sci. Fenn. Ser. A I Math.* 18 (1993), 77-91.
- [30] J.-M. Wu, Hausdorff dimension and doubling measures on metric spaces. *Proc. Amer. Math. Soc.* 126 (1998), 1453-1459.

DEPARTMENT OF MATHEMATICAL SCIENCES, P.O. BOX 3000, 90014 UNIVERSITY OF OULU, FINLAND

E-mail address: `changhao.chen@oulu.fi`

DEPARTMENT OF MATHEMATICS, HUBEI UNIVERSITY, WUHAN 430062, CHINA

E-mail address: `sywen_65@163.com`